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https://tinyurl.com/QC-WS23

Quantencomputing und Quantensimulation
Wintersemester 2023 - Übungsblatt 10
Ausgabe: 19.01.2024, Abgabe: 26.01.2024, Übungen: 29.01.2024

Aufgabe 24: Jordan-Wigner Transformation (8 points)

In the lecture, the Jordan-Wigner transformation was introduced to transform fermionic systems into spin systems, which can be simulated on a quantum computer. The vacuum state (the state without particles) is simulated as a spin up state and the one-particle state as a spin down state,

$$|0\rangle = a\,|1\rangle \equiv |\uparrow\rangle\,, \quad |1\rangle = a^\dagger\,|0\rangle \equiv |\downarrow\rangle\,.$$

At first sight, this gives us an equivalence between the ladder operators of the spin states $\sigma_\pm = 1/2(\sigma_x \pm i\sigma_y)$ and the creation and annihilaton operators of the fermions $a^{(\dagger)}$,

$$a \equiv \sigma_+, \quad a^\dagger \equiv \sigma_-.$$

a) (1 point) Use $[\sigma_+, \sigma_-] = \sigma_z$ to show $1 - 2a^\dagger a \equiv \sigma_z$.

Let us now consider fermions on a chain. This shows that the above equivalence no longer works, as the creation and annihilation operators for different lattice sites ($i \neq j$) has to anti-commute, however for different lattice sites the ladder operators commute,

$$a_i^\dagger a_j^\dagger = -a_j^\dagger a_i^\dagger \quad \leftrightarrow \quad \sigma_{i,-}\sigma_{j,-} = \sigma_{j,-}\sigma_{i,-}.$$

In order to express this property using the spin operators, the transformation

$$a_i \to \sigma_{i,+} \otimes_{k=1}^{i-1} \sigma_{k,z}, \quad a_i^\dagger \to \sigma_{i,-} \otimes_{k=1}^{i-1} \sigma_{k,z}$$

is performed. We therefore accumulate a phase of -1 per spin down state on the lattice sites before i.
b) (2 points) Show that this preserves the canonical anti-commutation relations $\{a_i, a_j^\dagger\} = \delta_{i,j}$ and $\{a_i, a_j\} = 0$.

c) (1 point) Show that $a_n^\dagger a_{n+1} \equiv \sigma_{n,-}\sigma_{n+1,+}$.

d) (1 point) Use the Jordan-Wigner transformation on the following fermionic Hamiltonian

$$H = \sum_n J_z \left(1 - 2(a_n^\dagger a_n + a_{n+1}^\dagger a_{n+1}) - 4a_n^\dagger a_{n+1}^\dagger a_n a_{n+1}\right) + \frac{J_\perp}{2}\left(a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n\right)$$

to obtain

$$H = \sum_n J_z \sigma_{n,z}\sigma_{n+1,z} + \frac{J_\perp}{2}\left(\sigma_{n,-}\sigma_{n+1,+} + \sigma_{n,+}\sigma_{n+1,-}\right).$$

The Jordan-Wigner transformation uses the occupation number representation. An alternative formulation is given by using the so-called parity basis. In the parity basis, the state of the ith qubit q_i is given by the formula $q_i = \sum_{j<i} f_j \mod 2$, where f_i describes the state of the ith qubit in the occupation number representation ($f_i = 1$ if a fermion occupies lattice site i). This alternative transformation is obtained as

$$a_i \to \otimes_{k=0}^{M-i-1} \sigma_{M-k,x}\sigma_{i,x}\sigma_{i-1,z} + i \otimes_{k=0}^{M-i-1} \sigma_{M-k,x}\sigma_{i,y},$$
$$a_i^\dagger \to \otimes_{k=0}^{M-i-1} \sigma_{M-k,x}\sigma_{i,x}\sigma_{i-1,z} - i \otimes_{k=0}^{M-i-1} \sigma_{M-k,x}\sigma_{i,y}.$$

d) (1 point) Write the state $|10100111\rangle$ given in the occupation number representation in the parity basis.

e) (2 points) Show that this transformation also preserves the canonical anti-commutation relations $\{a_i, a_j^\dagger\} = \delta_{i,j}$ and $\{a_i, a_j\} = 0$.

Aufgabe 25: Simulation of a one dimensional particle (3 points)

We consider the simulation of a one dimensional particle with the Hamiltonian

$$H = \frac{1}{2}p^2 + V(x).$$

In order to simulate this Hamiltonian with a circuit, we discretize space by using the (dimensionless) position operator $x = \sum_x x\,|x\rangle\langle x|$ and the approximated (dimensionless) momentum operator $p = -\frac{i}{2}\sum_x (|x+1\rangle\langle x| - |x-1\rangle\langle x|)$ (x here represents a binary number and $x\Delta x = xL/2^n$ describes the actual location). A general state $|\psi\rangle$ can then be described by $|\psi\rangle = \sum_x \psi(x\Delta x)\,|x\rangle$, where $\psi(x\Delta x)$ represents the wave function.
a) (1 point) Given the state $|p\rangle = U_{\mathrm{QFT}}\,|x\rangle$ with $U_{\mathrm{QFT}} = \frac{1}{\sqrt{2^n}}\sum_y e^{2\pi i x y/2^n}\,|y\rangle\langle x|$. Calculate $p\,|p\rangle$ for both the exact operator $p = -i\partial_x$ and its discretized approximation and compare the results.

b) (2 points) To implement the effect of the potential, the unitary mapping

$$U_V:\ |x\rangle\,|y\rangle \to |x\rangle\,|y \oplus \Delta t V(x)\rangle$$

is used. Show that the circuit shown below produces the state

$$U_V\,|\psi(0)\rangle\,U_{\mathrm{QFT}}^\dagger\,|1\rangle = \sum_{x=1}^{2^n-1} \langle x|\psi(0)\rangle\, e^{-2\pi i \Delta t V(x)/2^t}\,|x\rangle\,U_{\mathrm{QFT}}^\dagger\,|1\rangle\,.$$